Let $\rho = R\epsilon^z$. We then have, after some simplification,

$$\frac{\dfrac{d^2z}{d\theta^2}}{1+\left(\dfrac{dz}{d\theta}\right)^2} = \frac{\epsilon^{-z}+\epsilon^z}{\epsilon^{-z}-\epsilon^z}.$$

Whence, multiplying both sides by $2dz$, and integrating, we have

$$\log\left[1+\left(\frac{dz}{d\theta}\right)^2\right] = \log\frac{C_1^2}{(\epsilon^{-z}-\epsilon^z)^2}.$$

Hence,

$$1+\left(\frac{dz}{d\theta}\right)^2 = \frac{C_1^2}{(\epsilon^z-\epsilon^{-z})^2}$$

and

$$\frac{dz}{d\theta} = \pm\frac{\sqrt{C_1^2-(\epsilon^z-\epsilon^{-z})^2}}{\epsilon^z-\epsilon^{-z}}.$$

Using the + sign, we have

$$d\theta = \frac{(\epsilon^z-\epsilon^{-z})dz}{\sqrt{C_1^2-(\epsilon^z-\epsilon^{-z})^2}} = \frac{(\epsilon^z-\epsilon^{-z})dz}{\sqrt{C_1^2+4-(\epsilon^z+\epsilon^{-z})}}.$$

Whence, on integration, we have

$$\theta + C_2 = \sin^{-1}\left(\frac{\epsilon^z+\epsilon^{-z}}{\sqrt{C_1^2+4}}\right),$$

or

$$\epsilon^z+\epsilon^{-z} = \sqrt{C_1^2+4}\sin(\theta+C_2) = 2C\sin(\theta+C_2),$$

where

$$2C = \sqrt{C_1^2+4}.$$

Replacing ϵ^z by ρ/R and simplifying, we have

$$\rho^2 - 2CR\rho\sin(\theta+C_2) + R^2 = 0.$$

Transforming to rectangular coördinates, we have

$$x^2+y^2-2CR\sin C_2x - 2CR\cos C_2y + R^2 = 0,$$

the equation of a circle orthogonal to the circle $x^2+y^2=R^2$.

380. Proposed by C. N. SCHMALL, New York City.

Show that

$$\int_0^\infty\left[\frac{1}{1^4+x^2}+\frac{1}{2^4+x^2}+\frac{1}{3^4+x^2}+\cdots\right]dx = \frac{\pi^3}{12}$$

where the series in the brackets is infinite.

Criticism on the solution published in the January, 1916, MONTHLY, page 23, by T. H. GRONWALL, New York City.

The solution as published is open to criticism, inasmuch as the integrability term by term of a uniformly convergent series is assumed for an *infinite* interval of integration. Now the familiar argument runs thus:

$$S(x) = \sum_1^\infty s_n(x) = \sum_1^N s_n(x) + R_N(x),$$

N an integer, $|R_N(x)| < \epsilon$ uniformly for $a \leqq x \leqq b$, ϵ arbitrarily small, and N sufficiently large ($N \geqq N_0(\epsilon)$); whence

$$\left|\int_a^b S(x)dx - \sum_1^N\int_a^b s_n(x)dx\right| = \left|\int_a^b R_N(x)dx\right| < \epsilon(b-a),$$

from which the term by term integrability of $S(x)$ follows when a and b are finite. The above inequality has, however, no sense when $b = \infty$, so that the term by term integrability in an infinite interval demands further investigation. If one does not want to invoke general theorems in this direction, such as are given, for instance, in Bromwich's *Infinite Series* or de la Vallee Poussin's *Cour d'Analyse*, one may proceed as follows in this problem.

Let

$$S(x) = \sum_1^\infty \frac{1}{n^4 + x^2}$$

be uniformly convergent for $x \geqq 0$; then the term by term integration in the finite interval $0 \leqq x \leqq a$ is legitimate:

$$\int_0^a S(x)dx = \sum_1^\infty \frac{1}{n^2} \arctan \frac{a}{n^2}.$$

Now arctan $(a/n^2) < (\pi/2)$ and all the terms of the right-hand series are positive; hence, for any $N \geqq 1$

$$\sum_1^N \frac{1}{n^2} \arctan \frac{a}{n^2} < \int_0^a S(x)dx < \frac{\pi}{2} \sum_1^\infty \frac{1}{n^2}.$$

Letting a tend toward infinity, *while keeping N fixed*, we find

$$\frac{\pi}{2} \sum_1^N \frac{1}{n^2} \leqq \int_0^\infty S(x)dx \leqq \frac{\pi}{2} \sum_1^\infty \frac{1}{n^2},$$

and finally letting N tend to infinity,

$$\int_0^\infty S(x)dx = \frac{\pi}{2} \sum_1^\infty \frac{1}{n^2} = \frac{\pi^3}{12}.$$

This point, that great caution should be exercised when inverting summation and integration in an infinite integration interval, or when the integrands may become infinite between the finite limits of integration, appears to me to be all the more important as the treatment of such questions in the current calculus textbooks is exceedingly inadequate.

393. Proposed by LAENAS G. WELD, Pullman, Ill.

Find the area of the least ellipse which can be drawn upon the face of a brick wall so as to inclose four bricks.

II. Solution by Frank R. Morris, Glendale, Calif.

In this solution the bricks are considered laid as in practical work.

Consider two of the bricks placed end to end in the same layer, a third in the layer above with its middle at the joint of the first two, and the fourth similarly placed in the layer below. Select as the origin the mid-point of the joint between the two bricks in the same layer and choose the x-axis parallel to the layers. Let the equation of the desired ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Since the four quadrants are symmetrical, it will be sufficient to consider the first one only. Call the upper, right-hand corner of the uppermost brick A and the corresponding corner of right-hand brick of the middle layer B. Let m represent the length of a brick and n its thickness. The ellipse will obviously pass through A or B or both. If it passes through A, whose coördinates are $(m/2, 3n/2)$, the equation is

$$\frac{\left(\frac{m}{2}\right)^2}{a^2} + \frac{\left(\frac{3n}{2}\right)^2}{b^2} = 1, \quad \text{or} \quad \frac{m^2}{a^2} + \frac{9n^2}{b^2} = 4.$$

Whence,

$$b = \frac{3an}{\sqrt{4a^2 - m^2}}, \quad \text{and the area } \pi ab \text{ is} \quad \frac{3\pi a^2 n}{\sqrt{4a^2 - m^2}},$$